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Letter to the Editor

On Rational Approximation to $|X|$

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Let $n \geqslant 1$ be an integer. We construct below real polynomials $P_n(x)$, $Q_n(x)$ of degree $\leqslant n$, so that

$$\left\| |x| - \frac{P(x)}{Q(x)} \right\| \leqslant (2n^2)^{-1},$$

where $\| \ \|$ is the uniform norm on $[-1, 1]$.

Let $T_{2n}(x)$ denote the Chebyshev polynomial (first kind) of degree $2n$. Then [1, p. 365] on [0, 1],

$$T_{2n}(x^{1/2}) = \sum_{j=0}^{n} (-1)^{n-j} 2^{2j} \frac{n}{n+j} \binom{n+j}{2j} x^j = P(x) - xQ(x),$$

where

$$P(x) \equiv \sum_{j=0}^{[n/2]} (-1)^{n-2j} 2^{4j} \frac{n}{n+2j} \binom{n+2j}{4j} x^{2j}$$

and

$$Q(x) \equiv \sum_{j=0}^{[(n-1)/2]} (-1)^{n-2j} 2^{4j+2} \frac{n}{n+2j+1} \binom{n+2j+1}{4j+2} x^{2j}.$$

Thus

$$\left\| |x| - \frac{P(x)}{Q(x)} \right\| = \left\| |x| - \frac{P(|x|)}{Q(|x|)} \right\|$$

$$= \max_{0 \leqslant x \leqslant 1} \left| x - \frac{P(x)}{Q(x)} \right|$$

0021-9045/89 $3.00

$$= \max_{0 \leqslant x \leqslant 1} \left| \frac{T_{2n}(x^{1/2})}{Q(x)} \right| \leqslant (2n^2)^{-1}$$

since, on $[0, 1]$, $|T_{2n}(x^{1/2})| \leqslant 1$ and $|Q(x)| \geqslant |Q(0)| = 2n^2$.

References

1. P. J. Davis, "Interpolation and Approximation," Dover, New York, 1975.
2. A. R. Reddy, Approximations to x^n and $|x|$—A survey, *J. Approx. Theory* **51** (1987), 127–137.